

Weil representations of $U(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$, $q > 3$ odd via presentation and compatibility of methods.

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Abstract

In this article we construct Weil representations of quasi-split unitary groups $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ ($q > 3$) attached to quadratic extensions of finite fields, by using Bruhat-like presentations of those groups. More precisely, we define Weil representations of $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ associating to each generator a linear map of a suitable \mathbb{C} -vector space satisfying the relations of the aforementioned presentation. In addition, we also address the natural question on the compatibility of our representation of $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ with the classical one constructed by Gérardin.

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1 Introduction

Nowadays, Weil representations are a vast topic in mathematical research. They arose from A. Weil's celebrated paper [22] as projective representations of the symplectic groups $\mathrm{Sp}(2n, F)$, where F is a locally compact field. In this work Weil took advantage of the representation theory of the Heisenberg group \mathcal{H}_n associated to a symplectic space of rank n , described by the Stone-von Neumann theorem for the real case. Weil representations have multiple consequences in diverse areas, including number theory, physics and algebra. A remarkable fact is that by decomposition into irreducible components, they provide a fruitful method for constructing representations of some classical groups over a finite or local field of residual characteristic

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different from two [10]. In the same way, in [13], [14], [12] the authors use Weil representations to give a description of the representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ and $\mathrm{SL}_2(\mathbb{Z}_p)$, including $p = 2$.

Regarding unitary groups, Gérardin [5] constructed Weil representations of unitary groups associated to unitary spaces (F, i) . More precisely, given a quadratic extension K/k of finite fields and $-$ the nontrivial k -automorphism of K , we let (F, i) denote a K -vector space F equipped with the nondegenerate skew-hermitian form i given by $-$. Let E be the underlying k -vector space of F . Gérardin attached to (F, i) a nondegenerate symplectic form j on E in such a way that the unitary group $\mathrm{U}(F, i)$ is embedded into the symplectic group $\mathrm{Sp}(E, j)$. Thus, he obtained a representation of $\mathrm{U}(F, i)$ by restricting the Weil representation of $\mathrm{Sp}(E, j)$ to the subgroup $\mathrm{U}(F, i)$.

In the seventies, Cartier observed that Weil representations of $\mathrm{Sp}(2, \mathbb{R})$ could be simply constructed by assigning to each generator of certain presentation of $\mathrm{SL}(2, \mathbb{R})$ a linear operator of $L^2(\mathbb{R})$ satisfying the relations of the given presentation. Then Soto-Andrade [20] considered this approach to construct a Weil representation of $\mathrm{Sp}(2n, F)$, where F is a finite field. For this purpose, he found a suitable presentation and the corresponding linear operators, regarding the symplectic group $\mathrm{Sp}(2n, F)$ as a sort of generalized special group $\mathrm{SL}(2)$,

Soto-Andrade's idea of considering $\mathrm{Sp}(2n, F)$ as an analogue of $\mathrm{SL}(2)$ was the seed for a subsequent work in which Pantoja and Soto-Andrade [15, 17] have used this approach for a more general setting. More precisely, they defined the groups $\mathrm{GL}_*^\varepsilon(2, A)$ and $\mathrm{SL}_*^\varepsilon(2, A)$ which are groups of matrices over an involutive ring $(A, *)$ whose entries satisfy certain commutation relations involving the involution $*$ and $\varepsilon = \pm 1$. For instance, if we take the full matrix ring $A = \mathrm{M}(n, F)$ and the involution $*$ given by matrix transposition then the corresponding group $\mathrm{SL}_*^{-1}(2, A)$ is the symplectic group $\mathrm{Sp}(2n, F)$ and $\mathrm{SL}_*^1(2, A)$ is the split orthogonal group $\mathrm{O}(n, n)(F)$.

Later on, Gutiérrez Frez, Pantoja and Soto-Andrade [8] gave a method to construct a very general Weil representation for $\mathrm{SL}_*^\varepsilon(2, A)$ when this group has a Bruhat-like presentation. Firstly, they define an abstract core data consisting of a module M over the involutive ring $(A, *)$ equipped with a suitable nondegenerate complex valued self pairing χ and its second order homogeneous companion γ . Then the authors associated to each generator of $\mathrm{SL}_*^\varepsilon(2, A)$ a suitable linear operator of \mathbb{C}^M involving the aforementioned data. Finally, they proved that those operators satisfy the commuting relations of the Bruhat-like presentation of $\mathrm{SL}_*^\varepsilon(2, A)$.

Since their origin Weil representations keep receiving much attention. Indeed, there are several papers concerning this topic. For instance, Tanaka [18, 19] constructed all irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ by using Weil representations, Soto-Andrade [20] obtained all the irreducible representations of $\mathrm{Sp}(4, F)$, by decomposing both Weil representations associated to the two isomorphism types of quadratic forms over F of rank 4, besides the previously mentioned Gerardin's work [5].

Some recent developments have been accomplished by Aubert and Przebinda [1], who gave a wide and detailed description of the Weil representation attached to a symplectic group over a finite or local field, providing explicit computations and formulae. Herman and Szechtman [4] constructed Weil representations of unitary groups attached to ramified quadratic extension of local rings (odd characteristic) by embedding these groups into a symplectic group. Dutta and Prasad [3] defined a Weil representation associated to a finite abelian group of odd order, proving that it is multiplicity free and that each irreducible component is associated to an element of a partially ordered set.

Regarding Weil representations via presentations, we may mention Vera-Gajardo [21] constructed a generalized Weil representation for the split orthogonal group $O(2n, 2n)$ over a finite field \mathbb{F}_q with $q > 3$, proving its compatibility with Howe's theory of dual pairs. Gutiérrez Frez [7] constructed a generalized Weil representation of $SL_*^\varepsilon(2, A)$, where A is the truncated polynomial ring $\mathbb{F}_q[x]/\langle x^m \rangle$ equipped with the \mathbb{F}_q -linear involution $x \mapsto -x$ for $\varepsilon = -1$, whereas Gutiérrez Frez and Pantoja [9] studied the case $\varepsilon = 1$.

Let $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ denote the group of isometries of a nondegenerate hermitian bilinear form on a \mathbb{F}_{q^2} -vector space of dimension $2n$. In this article, we construct a generalized Weil representation for the groups $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ for q odd greater than 3, regarding them as an $SL_*^\varepsilon(2, A)$ group for a suitable unitary ring A . For this purpose we will prove that the group has a Bruhat-like presentation, and then we will construct some data which allow us to obtain the desired representation.

The main results of the paper are summarized by the following theorems:

Theorem 3.1. The quasi-split unitary group $U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is isomorphic to $SL_*^{-1}(2, A_n)$, where A_n is the full matrix ring $M_n(\mathbb{F}_{q^2})$ equipped with the involution given by $(a^*)_{ij} = \overline{a_{ji}}$.

According to [16] Theorem 15, the group $SL_*^{-1}(2, A_n)$ has a Bruhat-like presentation. Therefore this fact and the result above will allow us to construct a generalized Weil representation for the group. In Theorem 4.4, we construct a data $(M, \chi, \gamma, -1/q^n)$ for $SL_*^{-1}(2, A_n)$ and by using Theorem 4.4 in [8] we prove the following result.

Theorem 4.4. There exists a linear (complex) representation $(\text{Aut}(\mathbb{C}^M), \rho)$ of $SL_*^{-1}(2, A_n)$ defined on the generators as follows:

- $\rho_{u_b}(e_a) = \gamma(b, a)e_a,$
- $\rho_{h_t}(e_a) = \alpha(t)e_{at^{-1}},$
- $\rho_w(e_a) = c \sum_{b \in M} \chi(-a, b)e_b,$

which we call generalized Weil representation of $SL_*^{-1}(2, A) \cong U(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ associated to the data $(M, \chi, \gamma, -1/q^n)$.

Furthermore we address the natural question on the compatibility of our representation with the classical one given by G  radin [5]. In fact, we prove:

Theorem 6.1. The restriction of the classical Weil representation of the symplectic group $\mathrm{Sp}(E, j)$ to the group $\mathrm{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$ is the representation (\mathbb{C}^M, ρ) constructed in Theorem 1.2.

Specifically, the article is organized as follows. In section 2, we introduce the main concepts on generalized classical groups $\mathrm{GL}_*^\varepsilon(2, A)$, $\mathrm{SL}_*^\varepsilon(2, A)$ and a construction of Weil representations of $\mathrm{SL}_*^\varepsilon(2, A)$ attached to a data. In section 3, we verify that the quasi-split unitary group $\mathrm{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$ has a Bruhat-like presentation. In section 4, we construct a data for the group $\mathrm{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$ and we write the linear operator associated to each generator explicitly. In section 5, we give an initial decomposition of the constructed representation. Finally in section 6, we prove the compatibility of methods for the group $\mathrm{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$ and we show that the components of the initial decomposition are irreducible.

2 Preliminaries

In this section, we shall introduce the concepts of generalized classical groups $\mathrm{GL}_*^\varepsilon(2, A)$ and $\mathrm{SL}_*^\varepsilon(2, A)$ attached to a unitary ring A endowed with an involution $*$. In addition, we also recall the method in [8] to construct generalized Weil representations of $\mathrm{SL}_*^\varepsilon(2, A)$ when a Bruhat-like presentation and a data for the group are provided.

2.1 Generalized Classical Groups

Let A be a unitary ring with an involution $a \mapsto a^*$, *i.e.* an antiautomorphism of the ring A of order two. Let A^\times be the group of invertible elements of A , $Z(A)$ the center of A and $A^{\varepsilon\text{-sym}}$ the set of all ε -symmetric elements of A , that is $A^{\varepsilon\text{-sym}} = \{a \in A \mid a^* = -\varepsilon a\}$. The involution on A induces an involution on the full matrix ring $\mathrm{M}(2, A)$, namely $(g^*)_{ij} = (g_{ji})^*$. For $\varepsilon = 1$ or $\varepsilon = -1$ in A we set

$$J = \begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix} \in \mathrm{M}(2, A).$$

We write $\mathrm{ML}_*^\varepsilon(2, A)$ to denote the set of matrices in $\mathrm{M}(2, A)$ such that $g^* J g J^{-1} = \delta(g) I$, for some $\delta(g) \in Z(A) \cap A^{\varepsilon\text{-sym}}$. In analogy to the classical case, Pantoja and Soto-Andrade [17] define a $*$ -determinant function by $\det_*(g) = ad^* + \varepsilon bc^*$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then they proved the following result.

Proposition 2.1. The set $\mathrm{GL}_*^\varepsilon(2, A)$, consisting of all invertible elements in $ML_*^\varepsilon(2, A)$, is a group under matrix multiplication and \det_* is an epimorphism of $\mathrm{GL}_*^\varepsilon(2, A)$ onto the group of all central ε -symmetric invertible elements of A .

Definition 2.2. The group $\mathrm{SL}_*^\varepsilon(2, A)$ is the kernel of the epimorphism \det_* .

Remark 2.3. Notice that $\mathrm{SL}_*^\varepsilon(2, A)$ is the group of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in A that satisfy the following equalities: $a^*c = -\varepsilon c^*a$, $ab^* = -\varepsilon ba^*$, $b^*d = -\varepsilon d^*b$, $cd^* = -\varepsilon dc^*$ and $ad^* + \varepsilon bc^* = a^*d + \varepsilon c^*b = 1$.

2.2 Weil Representation for $\mathrm{SL}_*^\varepsilon(2, A)$

Let A be a unitary ring with an involution $*$. We set

$$h_t = \begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix} (t \in A^\times), \quad w = \begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}, \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} (s \in A^{\varepsilon\text{-sym}})$$

Definition 2.4. Let A be a unitary ring equipped with an involution $*$. We will say that $\mathrm{SL}_*^\varepsilon(2, A)$ has a Bruhat-like presentation if it is generated by the above elements with defining relations:

1. $h_t h_{t'} = h_{tt'}$, $u_s u_{s'} = u_{s+s'}$;
2. $w^2 = h_\varepsilon$;
3. $h_t u_s = u_{tst^*} h_t$;
4. $wh_t = h_{t^{*-1}} w$;
5. $wu_{t^{-1}} w u_{-\varepsilon t} w u_{t^{-1}} = h_{-\varepsilon t}$, $t \in A^\times \cap A^{\varepsilon\text{-sym}}$.

Let us suppose that the involutive ring A is finite and that the group $G = \mathrm{SL}_*^\varepsilon(2, A)$ has a Bruhat-like presentation.

Definition 2.5. Let M be a finite right A -module and let us consider a bi-additive map $\chi : M \times M \longrightarrow \mathbb{C}^\times$, a linear character $\alpha \in \widehat{A}^\times$, a function $\gamma : A^{\varepsilon\text{-sym}} \times M \longrightarrow \mathbb{C}^\times$ and a complex number c . We will say that (M, χ, γ, c) is a data for $\mathrm{SL}_*^\varepsilon(2, A)$ if the following properties hold:

1. For all $x, y \in M, t \in A^\times$,

(a) $\chi(xt, y) = \alpha(tt^*)\chi(x, yt^*)$.

(b) $\chi(y, x) = \chi(-\varepsilon x, y)$.

(c) $\chi(x, y) = 1$ for any $x \in M \Rightarrow y = 0$.

2. For all $s, s' \in A^{\varepsilon\text{-sym}}, x \in M$,

(a) $\gamma(s + s', x) = \gamma(s, x)\gamma(s', x)$.

(b) $\gamma(b, xt) = \gamma(tbt^*, x)$.

(c) $\gamma(t, x + z) = \gamma(t, x)\gamma(t, z)\chi(x, zt)$.

3. A complex number c such that $c^2|M| = \alpha(\varepsilon)$, and for all $s \in A^{\varepsilon\text{-sym}} \cap A^\times$ the following equality holds:

$$\sum_{y \in M} \gamma(s, y) = \frac{\alpha(\varepsilon s)}{c}.$$

We present now a theorem due to Gutiérrez Frez, Pantoja and Soto-Andrade which gives a method to construct Weil representations for $\text{SL}_*^\varepsilon(2, A)$ when a Bruhat-like presentation and a data (M, χ, γ, c) for the group are provided.

Theorem 2.6. Let us assume that $\text{SL}_*^\varepsilon(2, A)$ has a Bruhat-like presentation. If (M, χ, γ, c) is a data for the group then there exists a (linear) representation (\mathbb{C}^M, ρ) of $\text{SL}_*^\varepsilon(2, A)$ defined on the generators by:

1. $\rho_{u_b}(e_a) = \gamma(b, a)e_a$,

2. $\rho_{h_t}(e_a) = \alpha(t)e_{at^{-1}}$,

3. $\rho_w(e_a) = c \sum_{b \in M} \chi(-\varepsilon a, b)e_b$,

for $a \in M$, $b \in A^{\varepsilon\text{-sym}}$, $t \in A^\times$ and e_a the Dirac delta function at a , defined by $e_a(u) = 1$ if $u = a$ and $e_a(u) = 0$ otherwise.

This representation is called generalized Weil representation of $\text{SL}_*^\varepsilon(2, A)$ associated to the data (M, χ, γ, c) .

3 The group $\text{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$

In what follows we introduce the unitary group $\text{U}(n, n) (\mathbb{F}_{q^2}/\mathbb{F}_q)$ attached to a unitary space over a finite field. We also show that this group can be regarded as a generalized classical group $\text{SL}_*^{-1}(2, A)$ for a suitable involutive ring defined in the previous section.

Let $K = \mathbb{F}_{q^2}$ be a quadratic extension of the finite field $k = \mathbb{F}_q$, where q is odd. Let us denote $\lambda \mapsto \bar{\lambda}$ the nontrivial k -automorphism of K . For $y = (y_1, y_2, \dots, y_l) \in K^l$, we put $y^* = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_l)^t$, where t denotes the transposition.

We consider the K -vector space $F = K^{2n}$ whose elements are row vectors. We define the hermitian form $h : F \times F \rightarrow K$ given by $h(x, y) = xJ_+y^*$, where $J_+ = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$. Since all nondegenerate hermitian bilinear forms over a finite field are equivalent [6], we can see the quasi-split unitary group $U(n, n)(K/k)$ as the isometry group of the form h .

Henceforth we will use A_n to indicate the the full matrix ring $M_n(K)$ endowed with the involution $*$ defined by $(a^*)_{ij} = \overline{a_{ji}}$.

Theorem 3.1. The quasi-split unitary group $U(n, n)(K/k)$ is isomorphic to $SL_*^{-1}(2, A_n)$.

Proof. $SL_*^{-1}(2, A_n)$ is the group of isometries of the nondegenerate skew-hermitian form i given by $i(x, y) = xJ_-y^*$, where $J_- = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. According to [2] there is $\lambda \in K^\times$ such that $\lambda \cdot i$ is a nondegenerate hermitian form and therefore $\lambda \cdot i$ is equivalent to the form h above. Thus $SL_*^{-1}(2, A_n)$ is isomorphic to $U(n, n)(K/k)$. \square

According to [16] Theorem 15 and the result above, the group $U(n, n)(K/k)$ has a Bruhat-like presentation when $q > 3$. So we will construct a data for this group in order to obtain a generalized Weil representation of $U(n, n)(K/k)$.

4 Data for $SL_*^{-1}(2, A_n)$

The goal in this section is to construct a concrete data for the group $U(n, n)(K/k) = SL_*^{-1}(2, A_n)$. This will allow us to get a generalized Weil representation of $SL_*^{-1}(2, A_n)$ by using the method via presentation given in section two.

Let V be the K -vector space given by K^n . We consider the pairing $\langle \ , \ \rangle : V \times V \rightarrow K$ given by:

$$\langle u, v \rangle = uv^*.$$

We observe directly that this pairing is nondegenerate. We also see that $\langle xa, y \rangle = \langle x, ya^* \rangle$, for any $a \in A_n^\times$.

Let $M = V$ be the right A_n -module with the A_n -action given by the right multiplication. Let us denote by sgn the unique nontrivial character of K^\times whose square is trivial. Let ψ be a nontrivial character of K^+ such that $\psi(\lambda) = \psi(\bar{\lambda})$ for all $\lambda \in K^+$.

We consider the bi-additive function $\chi : M \times M \longrightarrow \mathbb{C}^\times$ given by $\chi((x, y)) = \psi(\langle x, y \rangle)$ and the linear character of A_n^\times given by

$$\alpha(t) = \text{sgn}(\det(t)) \text{ for } t \in A_n^\times.$$

We show then the following result.

Proposition 4.1. The map χ satisfies the conditions:

1. $\chi((xa, y)) = \alpha(aa^*)\chi((x, ya^*))$, for any $x, y \in M$ and any $a \in A_n$.
2. $\chi((y, x)) = \chi((x, y))$, for any $x, y \in M$.
3. χ is nondegenerate.

Proof. From definition of α we obtain $\alpha(aa^*) = 1$ for every $a \in A_n$. For $x, y \in M$ and $a \in A_n$, we see that,

1.

$$\chi((xa, y)) = \psi(\langle xa, y \rangle) = \psi(\langle x, ya^* \rangle) = \chi((x, ya^*)).$$

2.

$$\chi((y, x)) = \psi(\langle y, x \rangle) = \psi(yx^*) = \psi((xy^*)^*) = \psi(\langle x, y \rangle).$$

3. Finally, we prove that χ is nondegenerate. Suppose that $\chi((x, y)) = 1$ for all $x \in M$. If $y \neq 0$, we observe that the linear functional $\langle \cdot, y \rangle : V \longrightarrow K$ is nontrivial which implies it is surjective. We consider then $\lambda \in K$ satisfying $\psi(\lambda) \neq 1$, and $t \in M$ such that $\lambda = \langle t, y \rangle$. Then we get

$$1 = \psi(\langle t, y \rangle) = \psi(\lambda) \neq 1,$$

which forces $y = 0$.

□

The corresponding properties for γ are given by,

Proposition 4.2. Let $\gamma : A_n^{\varepsilon\text{-sym}} \times M \longrightarrow \mathbb{C}^\times$ the function given by $\gamma(u, x) = \psi(2^{-1}\langle xu, x \rangle)$. Then,

1. $\gamma(u + u', x) = \gamma(u, x)\gamma(u', x)$, for $x \in M$ and $u, u' \in A_n^{\varepsilon\text{-sym}}$.
2. $\gamma(u, xa) = \gamma(aua^*, x)$, for $x \in M$, $u \in A_n^{\varepsilon\text{-sym}}$ and $a \in A_n^\times$.
3. $\gamma(u, x + y) = \gamma(u, x)\gamma(u, y)\chi(x, yu)$, for $x, y \in M$ and $u \in A_n^{\varepsilon\text{-sym}}$.

Proof. Let $u, u' \in A_n^{\varepsilon\text{-sym}}, a \in A_n^\times, x, y \in M$,

1. From additivity of $\langle \cdot, \cdot \rangle$, we see that

$$\gamma(u + u', x) = \psi(2^{-1}(\langle xu + xu', x \rangle)) = \psi(2^{-1}\langle x, xu \rangle)\psi(2^{-1}\langle x, xu' \rangle) = \gamma((u, x))\gamma((u', x)).$$

2. By Proposition 4.1 part 2, we have

$$\gamma(u, xa) = \psi(2^{-1}\langle xau, xa \rangle) = \psi(2^{-1}\langle xaua^*, x \rangle) = \gamma((aua^*, x)).$$

3. From definition we see

$$\gamma(u, x + y) = \psi(2^{-1}(xu + yu)(x + y)^*) = \gamma((u, x))\gamma((u, y))\psi(2^{-1}(xuy^* + yux^*))$$

Since ψ is an additive character of K and it is $*$ -invariant, we obtain

$$\psi(2^{-1}(xuy^* + yux^*)) = \psi(2^{-1}xuy^*)\psi(2^{-1}yux^*) = \psi(2^{-1}xuy^*)\psi(2^{-1}(x(yu)^*)^*) = \psi(xuy^*) = \chi(x, yu).$$

□

For any invertible symmetric element u in $A_n^{-1-\text{sym}}$, we define $Q_u : K^n \rightarrow k$ given by $Q_u(x) = \langle xu, x \rangle$.

Then we prove,

Lemma 4.3. The map Q_u is a nondegenerate and nonsplit k -quadratic form on K^n . Furthermore, the forms $Q_u, Q_{u'}$ are equivalent for any $u, u' \in A_n^\times \cap A_n^{\varepsilon\text{-sym}}$.

Proof. We observe directly that $Q_u(\lambda x) = \lambda^2 Q_u(x)$ for any $\lambda \in k$ and $x \in K^n$. Now, let us consider the form B_u associated to Q_u .

$$B_u(x, y) = Q_u(x + y) - Q_u(x) - Q_u(y) = \langle x, yu \rangle + \overline{\langle x, yu \rangle}$$

It is clear that B_u is a symmetric bilinear form. To prove that B_u is nondegenerate, let us suppose that $B_u(x, y) = 0$ for all $x \in K^n$. Then, for every $x \in K^n$, we have $\langle x, yu \rangle \in \{\lambda\Delta/\lambda \in k\}$. If $y \neq 0$, then the K -linear functional $\langle \cdot, yu \rangle : K^n \rightarrow K$ is surjective. Thus for any $t \in k$, there exists $x_0 \in K^n$ such that $\langle x_0, yu \rangle = t$ which is impossible and then $y = 0$.

Finally, let u, u' be two elements in $A_n^\times \cap A_n^{\text{sym}}$. Since all nondegenerate hermitian forms in a finite field

are equivalent and u and u' are invertible hermitian matrices, there exists $j \in A_n^\times$ such that $juj^* = u'$. Thus we obtain $Q_{u'}(x) = Q_u(jx)$. \square

Notice that Q_{id} is the quadratic form given by the sum of the field norm of each coordinate. In order to facilitate calculations, we will consider only Q_{-id} and denote it by Q .

We keep considering the above notations and set

$$c = \frac{-1}{q^n}.$$

Then we should prove

$$\sum_{x \in K^n} \psi(\langle x, x \rangle) = \frac{\alpha(-(-id))}{c} = \frac{1}{c}$$

Indeed, the above sum is a Gauss sum corresponding to this quadratic form on a vector space of dimension q^{2n} over k . Therefore the sum is $-q^n$ (see [20]), which is the desired result.

Propositions 4.1, 4.2 and Lemma 4.3 show that:

Theorem 4.4. There exists a linear representation $(\text{Aut}(\mathbb{C}^M), \rho)$ of $\text{SL}_*^{-1}(2, A_n) \cong \text{U}(n, n)(K/k)$ defined on the generators as follows:

- $\rho_{u_b}(e_a) = \gamma(b, a)e_a,$
- $\rho_{h_t}(e_a) = \alpha(t)e_{at^{-1}},$
- $\rho_w(e_a) = c \sum_{b \in M} \chi(-a, b)e_b,$

for $a \in M$, $b \in A_n^{\varepsilon\text{-sym}}$, $t \in A_n^\times$ and e_a the Dirac delta function at a , defined by $e_a(u) = 1$ if $u = a$ and $e_a(u) = 0$ otherwise. This representation is called the generalized Weil representation of $\text{SL}_*^{-1}(2, A) \cong \text{U}(n, n)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ associated to the data $(M, \chi, \gamma, -1/q^n)$.

Proof. We see that propositions 4.1, 4.2 and lemma 4.3 proves that (M, χ, γ, c) is a data for the group $\text{SL}_*^{-1}(2, A_n)$. Thus according to [8] Theorem 4.4 we finally obtain the desired representation. \square

5 Initial Decomposition

In this section we obtain an initial decomposition of the Weil representation (\mathbb{C}^M, ρ) of $G = \text{SL}_*^\varepsilon(2, A)$. In order to do this, we take advantage of the fact that there is a group of intertwining operators that acts

naturally in \mathbb{C}^M , namely the unitary group $U(\chi, \gamma)$ of the pair (χ, γ) . To this purpose, we lean on [8] Theorem 7.6.

At the end of this section we draw on this general decomposition for our particular case $U(n, n)(K/k)$.

Definition 5.1. The unitary group $U(\chi, \gamma)$ of the pair (χ, γ) is the group consisting of all A_n -linear automorphism φ of M such that $\gamma(b, \varphi(x)) = \gamma(b, x)$, for any $b \in A_n$ and $x \in M$.

In what follows, we just put U for $U(\chi, \gamma)$.

Following the idea of [8], if we know the structure of the group U and the set of its irreducible representations, we can find an *initial* decomposition of the Weil representation in the sense that the obtained components are not necessarily irreducible. In the sequel, we make this decomposition explicit.

The group U acts (naturally) on \mathbb{C}^M via $f(\beta.x)$, where $\beta.x = \beta(x)$, for $\beta \in U$ and $x \in M$. It follows that U acts on $Aut_{\mathbb{C}}(\mathbb{C}^M)$, explicitly we have

$$\begin{aligned}\sigma : U &\longrightarrow Aut_{\mathbb{C}}(\mathbb{C}^M), \\ \sigma_{\beta}(f)(x) &= f(\beta^{-1}.x).\end{aligned}$$

Arguing as in [8] we see the natural action of U on \mathbb{C}^M commutes with the action of the Weil representation of $G = SL_{*}^{\varepsilon}(2, A)$.

Let \widehat{U} be the set of the irreducible representations of U . We consider the isotypic decomposition of \mathbb{C}^M with respect to U :

$$\mathbb{C}^M \cong \bigoplus_{(V_{\pi}, \pi) \in \widehat{U}} n_{\pi} V_{\pi}.$$

Since $n_{\pi} = \dim_{\mathbb{C}}(\text{Hom}_U(V_{\pi}, \mathbb{C}^M)) = \dim_{\mathbb{C}}(\text{Hom}_U(\mathbb{C}^M, V_{\pi}))$, if we put $m_{\pi} = \dim_{\mathbb{C}}(V_{\pi})$ we can write this decomposition in the following way:

$$\mathbb{C}^M \cong \bigoplus_{(V_{\pi}, \pi) \in \widehat{U}} m_{\pi} \text{Hom}_U(\mathbb{C}^M, V_{\pi}.)$$

For $(V_{\pi}, \pi) \in \widehat{U}$ and $\beta \in U$, we denote by π_{β} the map $\pi(\beta) : V_{\pi} \longrightarrow V_{\pi}$. The space $\text{Hom}_U(\mathbb{C}^M, V_{\pi})$ is formed by linear functions $\Theta : \mathbb{C}^M \longrightarrow V_{\pi}$ such that for any $\beta \in U$

$$\Theta \circ \sigma_{\beta} = \pi_{\beta} \circ \Theta. \tag{1}$$

Let us consider the Delta functions $\{e_x \mid x \in M\}$ and the map $\theta : M \longrightarrow V_{\pi}$ such that $\theta(x) = \Theta(e_x)$ for

all $x \in M$. Since $\sigma_\beta(e_x) = e_{\beta.x}$, condition (1) becomes:

$$\theta(\beta.x) = \pi_\beta \circ \theta(x). \quad (2)$$

Conversely, let $\theta : M \longrightarrow V_\pi$ satisfying (2). We extend linearly and we get a map $\Theta : \mathbb{C}^M \longrightarrow V_\pi$ such that (1) holds.

Thus, we can see the space $\text{Hom}_U(\mathbb{C}^M, V_\pi)$ as the function space whose elements are maps $\theta : M \longrightarrow V_\pi$ such that $\theta(\beta.x) = \pi_\beta \circ \theta(x)$, for all $\beta \in U$ and $x \in M$. The group $G = \text{SL}_*^\varepsilon(2, A)$ acts on this space via the Weil representation, using the same formulae as defined in Theorem (2.6). Similarly, it is possible to define the natural action of the group U in this space, as it is formed by functions with domain M .

Let ρ denote the Weil action of G on \mathbb{C}^M and $\hat{\rho}$ the Weil action of G on $\bigoplus_{(V_\pi, \pi) \in \hat{U}} m_\pi \text{Hom}_U(\mathbb{C}^M, V_\pi)$.

Due to the definition of the Weil representation, there exist scalars $K_g(x, y) \in \mathbb{C}$ depending only on $g \in G$ and $x, y \in M$ such that for all $f \in \mathbb{C}^M$, $\Lambda \in \bigoplus_{(V_\pi, \pi) \in \hat{U}} m_\pi \text{Hom}_U(\mathbb{C}^M, V_\pi)$ the following statements hold:

$$\rho_g(f) = \sum_{y \in M} K_g(\cdot, y) f(y); \quad (3)$$

$$\hat{\rho}_g(\Lambda) = \sum_{y \in M} K_g(\cdot, y) \Lambda(y). \quad (4)$$

Hence, we get:

Lemma 5.2. (\mathbb{C}^M, ρ) and $\left(\bigoplus_{(V_\pi, \pi) \in \hat{U}} m_\pi \text{Hom}_U(\mathbb{C}^M, V_\pi), \hat{\rho} \right)$ are isomorphic representations of G .

Finally we have (see [21]).

Proposition 5.3. The space $\text{Hom}_U(\mathbb{C}^M, V_\pi)$ is invariant under the Weil action of G .

Having made the decomposition above explicit, our aim is to obtain an initial decomposition for our particular case $\text{SL}_*^{-1}(2, A_n)$. For this purpose it is enough to know the structure of the group U and the set of irreducible representations, which is achieved in the next proposition.

Proposition 5.4. The unitary group $U = U(M, \chi, \gamma, c)$ is isomorphic to the center of $U(n, n)(K/k)$, namely, the set of $\lambda \in K$ such that $N_{K/k}(\lambda) = 1$.

Proof. Let β be an element in U . From definition β is K -linear, then $\beta(xa) = \beta(x)a$, for all $x \in K^n$ and $a \in A_n$. So in matrix language $\beta a = a\beta$, for all $a \in A_n$. This implies that β is a scalar transformation λid of K^n . Now the condition $\gamma(b, \varphi(x)) = \gamma(b, x)$ forces that λ has norm 1. \square

6 Compatibility of methods

In this last section, we address the natural question about the compatibility of methods to construct Weil representations of unitary groups. Specifically, we show that our generalized Weil representation of $U(n, n)(K/k)$ via presentation is equivalent to the classical one constructed by G  rardin [5].

Let us recall that the nondegenerate skew-hermitian form i on the $2n$ dimensional K -vector space F is given by $i(x, y) = xJ_-y^*$, where $J_- = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We denote by E the underlying k -vector space F (as in [5]), and consider the nondegenerate symplectic form j on E given by:

$$j(x, y) = i(x, y) - i(y, x) \quad x, y \in F.$$

Then the isometry group of i , which is $SL_*^{-1}(2, A_n) \cong U(n, n)(K/k)$, can be embedded into $Sp(E, j)$.

Hence we prove the following

Theorem 6.1. The restriction of the Weil representation of the symplectic group $Sp(E, j)$ to the group $SL_*^{-1}(2, A_n) \cong U(n, n)(K/k)$ is the representation (\mathbb{C}^M, ρ) constructed in Theorem 4.4.

Proof. We consider the following complete polarization of the space (K^{2n}, i) : F^- consists of all elements $x = (x_l) \in K^{2n}$ such that $x_l = 0$, for an $l > n$ and F^+ consists of all elements $x = (x_l) \in K^{2n}$ such that $x_l = 0$, for an $l < n$. Let $(\mathbb{C}^{F^-}, W^{(F, i)})$ be the restriction of the classic Weil representation of $Sp(E, j)$ to $SL_*^{-1}(2, A_n)$. It suffices to show that $W^{(F, i)}$ coincides with ρ on the generators of $SL_*^{-1}(2, A_n)$. To prove that, we will use Propositions 34, 35 and 36 in [1].

Let us note that the module M in section 4 is canonically isomorphic to F^- . Thus, we identify $x \in M$ with $(x, 0) \in F^-$.

We take a nontrivial character φ of the additive group K^+ such that $\varphi(-2\lambda) = \psi(\lambda)$ for all $\lambda \in K^+$ and recall that ψ is a nontrivial character of K^+ satisfying $\psi(\lambda) = \psi(\bar{\lambda})$ for all $\lambda \in K^+$. Then

1. Since h_t preserves F^+ and F^- we can use Proposition 34 in [1] to see that

$$W_{h_t}^{(F, i)}(f)(x) = s(\det h_t|_{F^-})f(xt).$$

Since $x \in F^-$, the action $x \mapsto xh_t$ is just $x \mapsto xt$. Then $\det h_t|_{F^-} = \det(t)$, whence $s(\det h_t|_{F^-}) = \alpha(t)$.

2. The element u_s acts trivially on F^+ and on K^{2n}/F^+ . Thus, from [1] Proposition 35 we get

$$W_{u_s}^{(F, i)}(f)(x) = \varphi(j(xc(-u_s), x))f(x).$$

We consider the Cayley transform $c(-u_s) = \begin{pmatrix} 0 & \frac{1}{2}s \\ 0 & 0 \end{pmatrix}$ in some basis. Then $j(xc(-u_s), x) = -\langle xs, x \rangle$ and hence $\varphi(j(xc(-u_s), x)) = \psi(2^{-1}\langle xs, x \rangle)$.

3. Notice that w maps bijectively F^- onto F^+ and F^+ onto F^- . So from the fact that $w^2 = -1$ and Proposition 36 in [1] we have

$$W_w^{(F,i)}(f)(x) = \gamma(1)^{\dim_k(X)} \frac{1}{|F^-|^{1/2}} \sum_{y \in F^-} \varphi(j(xw, y)) f(y)$$

We observe that $\gamma(1)^{\dim_k(X)} = 1$ since $\dim_k(X)$ is even and $\gamma(1)^2 = 1$. Then recalling that $\psi(\bar{\lambda}) = \psi(\lambda)$ for all $\lambda \in K$ we obtain,

$$\varphi(j(xw, y)) = \varphi(i(xw, y) - i(y, xw)) = \varphi(-2xy^*) = \psi(\langle x, y \rangle).$$

Therefore both representations of $U(n, n)(K/k)$ are equal, and the theorem follows.

□

According to Proposition 5.4, [5] Corollary 4.5 and the theorem above we get:

Corollary 6.2. The components of the generalized Weil representation (\mathbb{C}^M, ρ) of $U(n, n)(K/k)$ given in section 5 are irreducible.

References

- [1] Aubert, A., T. Przebinda. (2014). A reverse engineering approach to Weil Representation. *Cent. Eur. J. Math* 12 :1500–1585.
- [2] Dieudonné, J. (1967). *Sur les groupes classiques*. Publications de L’Institut de Mathématique de L’Université de Strasbourg, VI. Actualités Scientifiques et Industrielles, No. 1040. Hermann, Paris.
- [3] Dutta, K., Prasad, A. (2015). Combinatorics of finite abelian groups and Weil representations. *Pacific J. Math.* 275 (2) :295–324.
- [4] Herman, A., Szechtman, F. (2014). Weil representation of unitary groups associated to a ramified quadratic extension of local rings. *J. of Algebra.* 392: 158-184.
- [5] Gérardin P. (1977). Weil representations associated to finite fields. *J. Algebra.* 46:54-101.

- [6] Groove, L. (2002). *Classical groups and geometric algebra*. Graduate Studies in Mathematics, No. 39. American Mathematical Society.
- [7] Gutiérrez, L. (2009). A Generalized Weil Representation for $SL_*(2, A_m)$, where $A_m = \mathbb{F}_q[x]/\langle x^m \rangle$. *J. Algebra*. 332: 42–53.
- [8] Gutiérrez, L., Pantoja, J., Soto Andrade, J. (2011). On Generalized Weil Representations over Involutive Rings. *Contemp. Math*. 544: 109–122.
- [9] Gutiérrez, L., Pantoja, J. (2015). Weil representation of a generalized linear group over a ring of truncated polynomials ring over a finite field endowed with a second class of involution. *SIGMA*. 11, 076.
- [10] Kutzko, P. (1984). The exceptional representations of GL_2 . *Compos. Math*. 51: 3–14.
- [11] Lion, G., Vergne, M. (1980). *The Weil Representation, Maslov Index and Theta Series*. Progress in Mathematics, No. 6. Birkhäuser.
- [12] Nobs, A. (1977). Die irreduziblen Darstellungen von $GL_2(\mathbb{Z}_p)$, insbesondere $GL_2(\mathbb{Z}_2)$. *Math. Ann*. 229(2): 113–133.
- [13] Nobs, A. (1976). Die irreduziblen Darstellungen der Gruppen $SL_2(\mathbb{Z}_p)$, insbesondere $SL_2(\mathbb{Z}_2)$. I. *Comment. Math. Helv.* 51: 465–489.
- [14] Nobs, A., Wolfart, J. (1976) Die irreduziblen Darstellungen der Gruppen $SL_2(\mathbb{Z}_p)$, insbesondere $SL_2(\mathbb{Z}_2)$. II. *Comment. Math. Helv.* 51: 491–526.
- [15] Pantoja, J., Soto-Andrade, J. (2003). A Bruhat decomposition of the group $SL_*(2, A)$. *J. of Algebra*. 262:401–412.
- [16] Pantoja, J. (2006). A presentation of the group $SL_*(2, A)$, A a simple artinian ring with involution. *Manuscripta math.* 121: 97–104.
- [17] Pantoja, J., Soto-Andrade, J. (2009) Bruhat presentations for $*$ -classical groups. *Comm. in Algebra*. 37: 4170–4191.
- [18] Tanaka, S. (1967). Construction and classification of irreducible representations of special linear group of the second order over finite field. *Osaka J. Math.* 4:65–84.
- [19] Tanaka, S. (1967). Irreducible representations of the binary modular congruence groups $\mod p^\lambda$. *J. Math. Kyoto Univ.* 7: 123–132.

- [20] Soto-Andrade, J. (1978). Représentations de certain groupes symplectiques finis. *Bull. Soc. Math. France Mem.* 55-56: 5–334.
- [21] Vera-Gajardo, A., (2015). A Generalized Weil Representation for the finite split orthogonal group $O_q(2n, 2n)$, q odd, $q > 3$. *J. Lie Theory.* 25: 257–270.
- [22] Weil, A. (1964). Sur certains groupes d’opérateurs unitaires. *Acta Math.* 111: 143–211.